Spectral bounds for the number of Robin candidates

Eduardo Vega

Unreviewed preprint — version 1.2, revised October 27, 2026. Based on the September 15, 2026 circulation draft and local revision $v1.1$ of October 27, 2026. AI-assisted development and checking; not human peer reviewed.

Abstract

Let $R(X)$ count integers $3 \le n \le X$ with $\sigma(n) \ge \exp(\gamma_E) n \log\log n$. Under a stated zero-free region of width $\eta(t)$ and a uniform zero-density estimate for ζ, we prove $R(X) \le \exp(Lg(\log X))$ for $X \ge X_0$, with fixed $L > 0$ and X_0. The rate g is specified by a weight-two infimum $W(y) = \inf_{t \ge T_*}[\eta(t)\log y + 2\log t]$ and an explicit secondary factor. The proof combines a weighted sum over zeros, an exact centered Mertens identity, and a count of low-benefit prime-support patterns following classical benefit and partition methods. Classical and Vinogradov–Korobov inputs give unconditional specializations. In the classical case, the upper-bound exponent scale is proportional to $(\log X/\log\log X)\exp(-\sqrt{2a\log\log X})$, where $\eta(t) = a/\log t$. No numerical L or certified numerical X_0 is supplied. Prior implication by earlier work remains under investigation; no priority claim is made. The result does not prove or disprove RH, establish the existence of counterexamples, or give a global sign conclusion. This is an unreviewed preprint.

1 Main theorem and inputs

All logarithms are natural. Primes are ordinary positive rational primes; φ is Euler's totient, σ is the sum-of-divisors function, and γ_E is the Euler–Mascheroni constant. We count the weak reverse inequality, including equality, for $n \ge 3$. The finitely many small integers are retained throughout. The asymptotic symbols refer to the indicated real variable tending to ∞; constants may depend on fixed analytic inputs unless stated absolute.

Let η be a continuous, positive, nonincreasing zero-free function for ordinary ζ above a fixed $T_* > 1$, with $\eta \le 1/2$ and $\eta(t) \to 0$. Thus every nontrivial zero $\rho = \beta + i\gamma$ with $|\gamma| \ge T_*$ obeys $\beta \le 1 - \eta(|\gamma|)$. Admit a zero-density bound, with fixed $A_d > 0, B \ge 1, C \ge 0, C_d > 0, \sigma_0$ in $(1/2, 1)$,

$$N(\sigma, T) \le C_d T^{A_d(1-\sigma)^B}(\log T)^C$$

uniformly for $\sigma_0 \le \sigma < 1$ and $T \ge T_*$. Here $N(\sigma, T)$ counts zeros with $\beta > \sigma$ and $0 < \gamma < T$, with multiplicity; conjugate zeros are included separately when summing. Endpoint conventions can be replaced by one-sided limits in the integrals. Enlarge T_* once to meet the zero-free and density ranges.

Write, for y sufficiently large and $u = \log y$,

$$W(y) = W_2(y) = \inf_{t \ge T_*}\{\eta(t)u + 2\log t\},$$
$$P_0(y) = \left(\frac{u}{W(y)}\right)^{BC} \exp\left(\frac{A_d W(y)^{B+1}}{u^B}\right),$$
$$g(y) = \frac{y\sqrt{P_0(y)}e^{-W(y)/2}}{u}.$$

Assume $W = o(u)$ and $W/\log u \to \infty$. The first limit also follows from $\eta(t) \to 0$: fix t in the infimum, take limsup $W/u \le \eta(t)$, and then let t tend to ∞. Retain these exact definitions. Admit the usual ζ facts $0 < \beta < 1$, no real nontrivial zeros, the total zero count $N(T) = O(T \log T)$, the once-integrated explicit formula specified in Section 3.2, and the classical Mertens normalization. For the combinatorial step admit ordinary-prime Brun–Titchmarsh and ENS Lemmas 1–2 in the precise form stated in Section 2.3.

Define for real $X \ge 3$

$$R(X) = \#\{n \in \mathbb{Z} : 3 \le n \le X,\ \sigma(n) \ge e^{\gamma_E} n \log\log n\}.$$

The term "Robin candidate" here denotes exactly this counted set, not a claim that a counterexample exists. Robin's criterion [R84, Theorem 1] states that RH is equivalent to $\sigma(n) < \exp(\gamma_E) n \log\log n$ for every integer $n > 5040$. Consequently, for $X \ge 5040, R(X)$ is 26 plus the number of Robin counterexamples in $(5040, X]$; under RH that latter count is zero. The 26 small values are $3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520$ and 5040. This finite convention does not alter the asymptotic upper bound. The criterion is credited to Robin; related RH-conditional work of Ramanujan is described in [Ra97] .

Theorem 1. Under these inputs there exist fixed $L > 0$ and X_0 such that

$$R(X) \le \exp(Lg(\log X)) \qquad \text{for every } X \ge X_0. \tag{T1}$$

L and X_0 may depend on the fixed analytic inputs. No uniform numerical L or certified numerical X_0 is supplied. When the inputs are instantiated by their classical unconditional versions for ordinary zeta/primes, this is an unconditional written consequence of those theorems. The arbitrary η version remains explicitly conditional on the displayed input contract.

The additional imported statements are the primitive explicit formula in Section 3.2, the Mertens limit $\log M(y) - \log\log y \to \gamma_E$ in Section 4.2, and the prime-counting, prime-index and partition estimates in Section 2.3. Elementary Chebyshev and prime-power bounds are proved or reduced to finite sums below. References and the depth of the source checks are recorded in Section 7.

The proof has four separate steps: count a benefit relaxation for a positive budget H; construct H by spectral estimates; place all candidates in that relaxation by an exact identity; and aggregate full unit shells. The counting lemma alone constructs no H and says nothing about candidate inclusion. Neither inclusion nor counting uses a local θ-window error.

2 Counting a low-benefit relaxation

2.1 Definitions and counting proposition

Let y tend to $\infty, u = \log y$, and let $H(y) > 0$ eventually satisfy

$$\frac{\log H(y)}{\log y} \longrightarrow 0, \qquad H(y)(\log y)^3 \longrightarrow 0. \tag{P1}$$

Use $r(t) = \log(t/(t-1)), \lambda = r(y)/u, M(y) = \prod_{p \le y}(1 - 1/p)^{-1}, \theta(y) = \sum_{p \le y} \log p$, and the totient benefit

$$B_\varphi(n) = \log M(y) - \log \frac{n}{\varphi(n)} + \lambda(\log n - \theta(y)).$$

Let $E_y = E_y(H)$ be the positive integers in the unit log shell $[y-1, y]$ with $B_\varphi(n) \le H(y)$.

Proposition 2. There is an absolute $K_{\text{count}} > 0$ and an eventual threshold, allowed to depend on H, such that

$$|E_y| \le \exp(K_{\text{count}} G(y)), \qquad G(y) = y\sqrt{H(y)/u}. \tag{P2}$$

The same bound applies to the σ-benefit shell of Section 4.1, by $B_\sigma \ge B_\varphi$. This is an upper count of a relaxation. Section 4 supplies the separate candidate inclusion.

2.2 Repeated powers and distant support

Write $n = s * m$ with $s = \text{rad}(n)$. The nonnegative local costs imply

$$B_\varphi(n) = B_\varphi(s) + \lambda \log m.$$

Since $r(t)/\log t$ is strictly decreasing for $t > 1$, all support costs are nonnegative; further copies cost $\lambda \log p$ each. Thus the number of possibilities for m is at most $\exp(H/\lambda)$. Omitted primes $p < y/2$ each cost at least $1/(2y)$; their subsets have at most $(\text{floor}\, y + 1)^{\text{floor}(2yH)+1}$ possibilities. For included $p > 2y$,

$$r(p) \le \frac{1}{p-1} < \frac{1}{2y-1} \le \frac{2}{3}\lambda \log p,$$

so the product of these primes is at most $\exp(3H/\lambda)$. Integer size alone bounds the number of products by the same expression. Together these give an outside-choice count at most $\exp(F_{\text{out}})$, where

$$F_{\text{out}} = 4yuH + (2yH + 1)\log(y+1). \tag{P3}$$

For completeness, the omitted-prime estimate follows by setting $f(t) = r(t) - \lambda \log t$. Then $f(y) = 0$ and

$$-f'(t) = \frac{1}{t(t-1)} + \frac{\lambda}{t} \ge \frac{1}{y^2} \qquad (2 \le t \le y).$$

Integrating over $[y/2, y]$ gives $1/(2y)$. Also $1/\lambda \le yu$, as used in (P3). These arguments require only $y \ge 4$. They count enlargements of the permitted data; no claim of independent compatibility is needed for an upper bound.

2.3 Near support and imported prime estimates

All remaining choices are bits: whether a reference prime in $[y/2, y]$ is omitted, or a prime in $(y, 2y]$ is included. Set

$$d(p) = |r(p) - \lambda \log p|.$$

The derivative in Section 2.2 yields, on the entire interval $[y/2, 2y]$,

$$d(p) \geq \frac{|p-y|}{4y^2}. \tag{P4}$$

Use the disjoint lower interval $[y/2, y]$ and upper interval $(y, 2y]$. List each side's primes in order of increasing distance from y, calling them P_i with one-based indices, and put $h_i = |P_i - y|$. Free at most the first two bits on each side, paying a factor at most $16.A$ prime equal to y occurs only in the lower list and has distance zero; it is among these free bits.

For $i \geq 3$ put $j = i - 1 \geq 2$. On the upper side, $(y, P_i]$ contains i of the listed primes. On the lower side, $(P_i, y]$ contains $i - 1$, excluding P_i itself. Thus each interval contains at least j primes. Distinct integer locations also give $h_i \geq j \geq 2$. The imported Brun–Titchmarsh bound is $\pi(x + h) - \pi(x) \leq 2h/\log h$ for $x \geq 0$ and $h \geq 2$ [MV73; ENS, K]. The two applications have base points y and P_i respectively, both positive. Therefore

$$\begin{aligned} j &\leq \frac{2h_i}{\log h_i}, \\ h_i &\geq \frac{j \log j}{2} \geq 0.23 q_j \geq \frac{1}{5} q_j. \end{aligned} \tag{P5}$$

Indeed $h_i \geq j$ implies $\log h_i \geq \log j$, so the first inequality in (P5) immediately gives the second. Here q_j is the j-th ordinary prime, $q_1 = 2$ and $q_2 = 3$. The penultimate inequality is ENS Lemma 1 [ENS] , $j \log j \geq 0.46 q_j$ for $j \geq 2$. The argument is the same for the upper and lower lists; using $j = i - 1$ deliberately permits the lost endpoint in the lower list. The interval lengths here are at least 2, so no log-zero case is used.

For each side, let b_i be its changed-support bits after the first two. Since all costs are nonnegative, $(P4)-(P5)$ imply

$$\sum_{i \geq 3} q_{i-1} b_i \leq 20 y^2 H = T. \tag{P6}$$

Enlarge from bits to arbitrary nonnegative integer coefficients, and from the finite list to all ordinary primes. Let $\Pi(T)$ count all such prime partitions of total weight at most T, equivalently the finite-support sequences (x_j) of nonnegative integers with $\sum_j q_j x_j \leq T$. This is a finite family for $T \geq 0$: an occupied prime has $q_j \leq T$ and its coefficient is at most floor(T/q_j). The constrained bits inject into this family by assigning b_i to the distinct prime q_{i-1}; the free bits retain the omitted coordinates. Hence the finite near-pattern count is at most $16\Pi(T)^2$, before any asymptotic estimate is applied. Separately, ENS Lemma 2 [ENS] supplies

$$\log \Pi(T) = O\left(\sqrt{T/\log T}\right), \qquad T \longrightarrow \infty. \tag{P7}$$

There is no dependence on the budget restriction in ENS Theorem 6 : (P7) is a one-variable partition estimate, and the uniform interval estimate (P5) was applied directly to the actual primes. Both sides may use the whole budget H; that only enlarges the count.

2.4 Completion of the counting proof

The outside data, near bits, and m determine n uniquely: the omitted and included support slots determine rad(n), while m records all additional prime multiplicities. The slots for first presence and further copies are

disjoint. Allowing incompatible encoded data only enlarges this upper count. Thus the explicit intermediate estimate is

$$|E_y| \le 16e^{F_{\text{out}}}\Pi(20y^2H)^2. \tag{P8}$$

Under (P1), $T = y^{2+o(1)}$ tends to ∞ and $\log T \sim 2u$. Therefore $\sqrt{T/\log T} = O(G)$. Moreover

$$\frac{yuH}{G} = \sqrt{Hu^3} \longrightarrow 0, \qquad \frac{\log(y+1)}{G} \longrightarrow 0,$$

the latter since $G = y^{1+o(1)}$. The remaining outside term satisfies $2yH\log(y+1)/G = (2+o(1))\sqrt{Hu^3} \to 0$, by the same first limit. Hence $F_{\text{out}} = o(G)$, proving (P2) as written. In fact the encoding never uses shell membership, so it also bounds the full totient-benefit sublevel; the stated shell conclusion needs no such extension.

The partition input has only the variable T. In particular, its implied constant does not acquire hidden y-dependence when $T = 20y^2H(y)$. The full ENS integer-counting theorem is not invoked outside its printed budget range. The leading asymptotic for $\log\Pi(T)$ is $(2\pi/\sqrt{3})\sqrt{T/\log T}$; we use only its O bound and do not specify K_{count} numerically.

3 Constructing the spectral budget

Define $\theta(t) = \sum_{p\le t}\log p$ and $\psi(t) = \sum_{p^k\le t}\log p$. Let $\rho = \beta + i\gamma$ run over all nontrivial zeros with multiplicity, counting both signs of γ. Put

$$S(y) = \sum_{\rho}\frac{y^{\beta-1}}{|\rho(\rho-1)|}, \qquad M_z = \sum_{\rho}\frac{1}{|\rho(\rho-1)|} < \infty.$$

The total zero count implies convergence of M_z; the finitely many bounded height zeros have nonzero denominators. Each summand of S is decreasing in y. For $q > 0$ write $W_q(y) = \inf_{t\ge T_*}[\eta(t)u + q\log t]$, with $u = \log y$.

3.1 A split uniform in the moving parameter

For sufficiently small $0 < r < 1$, set

$$\delta = \left(\frac{r}{2A_d}\right)^{1/B}, \qquad \sigma_r = 1-\delta.$$

Then $\sigma_r \ge \sigma_0$ and the density exponent at σ_r equals $r/2$. Stieltjes integration, counting both signs of γ, gives

$$\begin{aligned}\sum_{\substack{\beta>\sigma_r\\ |\gamma|\ge T_*}}|\gamma|^{-r} &\le 2r\int_{T_*}^{\infty}N(\sigma_r,t)t^{-r-1}\,dt\\ &\le 2C_dr\int_1^{\infty}t^{-1-r/2}(\log t)^C\,dt = 2^{C+2}C_d\Gamma(C+1)r^{-C}.\end{aligned} \tag{U1}$$

The lower boundary contribution is nonpositive and can be dropped; the upper boundary vanishes since $N(\sigma_r, T)T^{-r} = O(T^{-r/2}(\log T)^C)$. Endpoint atoms can instead be handled with left limits, with the same bound. The γ integral is valid for every real $C \geq 0$, not just integer C. The constant in this estimate is independent of r; write $K_1 = 2^{C+2}C_d\Gamma(C+1)$.

Finitely many zeros below T_* lie to the left of σ_r for all sufficiently small r. For every zero with $\beta \leq \sigma_r$, including these finite low zeros,

$$\sum \frac{y^{\beta-1}}{|\rho(\rho-1)|} \leq M_z e^{-\delta u}, \qquad M_z = \sum_\rho \frac{1}{|\rho(\rho-1)|} < \infty.$$

For the remaining zeros, $|\rho(\rho-1)| \geq \gamma^2$, and the zero-free region gives

$$\begin{aligned}\frac{y^{\beta-1}}{|\rho(\rho-1)|} &\leq \exp(-[(1-\beta)u + (2-r)\log|\gamma|])\,|\gamma|^{-r} \\ &\leq e^{-W_{2-r}(y)}|\gamma|^{-r}.\end{aligned}$$

Sum this inequality using (U1). It proves

$$S(y) \leq M_z e^{-\delta u} + K_1 r^{-C} e^{-W_{2-r}(y)}. \tag{U2}$$

There is no substitution into a theorem with uncontrolled constants: (U2) is proved uniformly first.

Endpoint substitution.

Set

$$\delta = \frac{W}{u}, \qquad r = 2A_d(W/u)^B.$$

Since $W = o(u)$, all preceding small-r conditions hold eventually. Moreover, pointwise comparison inside the infimum gives

$$W_{2-r}(y) \geq (1 - r/2)W(y).$$

Indeed $\eta(t)u + (2-r)\log t$ is at least $(1-r/2)$ times $\eta(t)u + 2\log t$. Substitute this and $\delta u = W$ into (U2):

$$\begin{aligned}S(y) &\ll P_0(y)e^{-W(y)}, \\ P_0(y) &= \left(\frac{u}{W(y)}\right)^{BC} \exp\left(\frac{A_d W(y)^{B+1}}{u^B}\right).\end{aligned} \tag{U3}$$

The fixed factor $(2A_d)^{-C}$ is absorbed into the constant. Eventually $u/W > 1$, so $P_0 \geq 1$ and also absorbs the first term of (U2). In particular

$$\log P_0 = BC\log(u/W) + \frac{A_d W^{B+1}}{u^B} = o(W).$$

The result reaches the endpoint with a specified secondary loss. It does not in general show $S = O(\exp(-W))$.

3.2 The primitive formula and the integrated tail

We import the classical once-integrated explicit formula $[K;\ \text{GM},\ (1.3a)]$, normalized as

$$F(x) = \int_2^x (\psi(t) - t)\, dt$$
$$= -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\log(2\pi) + O(\log x).$$

The zero series is absolutely and uniformly convergent on compact intervals: the numerators are bounded there and the reciprocal quadratic denominators are summable. On a finite interval $[y, T]$, integration by parts followed by termwise integration therefore gives

$$\int_y^T \frac{\psi(t) - t}{t^2}\, dt = \sum_\rho \frac{y^{\rho-1} - T^{\rho-1}}{\rho(\rho-1)} + O(1/y).$$

The error is uniform in $T \geq y \geq 2$. The linear term contributes $O(1/y)$, and the primitive $O(\log x)$ remainder contributes $O(\log y/y^2)$. Since $\beta < 1$, the T-dependent zero sum tends to zero by dominated convergence, dominated by the summable reciprocal denominators. Thus the improper integral exists and

$$J_\psi(y) = \int_y^\infty \frac{\psi(t) - t}{t^2}\, dt = \sum_\rho \frac{y^{\rho-1}}{\rho(\rho-1)} + O(1/y),$$
$$|J_\psi(y)| \leq S(y) + C_J/y.$$

This avoids interchanging an unsmoothed zero series with an infinite integral. It also requires no reciprocal factor $(1-\beta)^{-1}$.

3.3 The weighted theta tail and the prime-power correction

Let $\Delta(t) = \theta(t) - t$, and define I_ψ and I_θ by integrating, respectively, $\psi(t) - t$ and $\Delta(t)$ against $a(t)/t^2$ below. Their existence will follow from the following argument.

The function $S(y) + C_J/y$ is a decreasing exact envelope: each zero has $\Re(\rho) < 1$ and the nonnegative series converges. Set

$$a(t) = \frac{1}{\log t} + \frac{1}{\log^2 t}.$$

Integration by parts first on a finite interval, then at ∞, gives

$$I_\psi(y) = a(y)J_\psi(y) + \int_y^\infty a'(t)J_\psi(t)\, dt,$$
$$|I_\psi(y)| \leq 2a(y)(S(y) + C_J/y). \tag{I8}$$

The endpoint $a(T)J_\psi(T)$ vanishes, and the integral against $|a'|$ converges by the decreasing envelope and $\int_y^\infty |a'| = a(y)$. This establishes existence of the weighted improper integral without requiring the majorant $P_0(t)\exp(-W(t))$ itself to be decreasing.

Let J_θ be the integral defining J_ψ with ψ replaced by θ. Its convergence follows from the first correction below. Keep the two prime-power corrections distinct:

$$Q_J(y) = J_\psi(y) - J_\theta(y) = \int_y^\infty \frac{\psi(t) - \theta(t)}{t^2}\, dt \geq 0,$$
$$Q_I(y) = I_\psi(y) - I_\theta(y) = \int_y^\infty \frac{(\psi(t) - \theta(t))a(t)}{t^2}\, dt \geq 0.$$

There are at most $\log(t)/\log(2)$ exponents $k \geq 2$ in $\psi - \theta$. Each inner sum over $p \leq t^{1/k}$ is at most $\sqrt{t}\log t$, by counting integers instead of primes. Hence $\psi(t)-\theta(t) = O(\sqrt{t}\log^2 t)$. This yields a convergent $Q_J = O(\log^2 y/\sqrt{y})$, and, using $a(t) \leq 2/\log t$ for $t \geq 4$,

$$Q_I = O(\log y/\sqrt{y}), \qquad I_\theta = I_\psi - Q_I. \tag{I9}$$

For example, $\int_y^\infty (\log t)t^{-3/2}dt = (2\log y + 4)/\sqrt{y}$. This proves existence of I_θ and controls its absolute value by $|I_\psi| + Q_I$. No $O(1/y)$ estimate for Q_J is used. A sharper $\Theta(1/\sqrt{y})$ statement for Q_J is separate and unnecessary here.

Now set $A(y) = P_0(y)\exp(-W(y))/u$. Then

$$\frac{\log y/\sqrt{y}}{A(y)} = \frac{u^2 e^{W-u/2}}{P_0} \longrightarrow 0,$$
$$\frac{1/(yu)}{A(y)} = \frac{e^{W-u}}{P_0} \longrightarrow 0.$$

Together with (I8), these yield $|I_\theta| \leq C_I A$ eventually for a fixed C_I after absorbing the prime-power and explicit-formula remainders. The fixed C_I includes these analytic remainders. The later choice of K_0 will leave only the two arithmetic corrections in Section 4 to absorb.

4 Centered inclusion of actual candidates

4.1 Benefits and the exact threshold

Retain $y \geq 4$ and $u = \log y$. For clarity, the benefit definitions are

$$\theta(y) = \sum_{p \leq y} \log p, \qquad \Delta(y) = \theta(y) - y,$$
$$r(t) = \log\frac{t}{t-1}, \qquad \lambda = r(y)/u, \qquad M(y) = \prod_{p\leq y}(1 - 1/p)^{-1},$$
$$B_\varphi(n) = \log M(y) - \log\frac{n}{\varphi(n)} + \lambda(\log n - \theta(y)),$$
$$B_\sigma(n) = \log M(y) - \log\frac{\sigma(n)}{n} + \lambda(\log n - \theta(y)).$$

Use the candidate convention

$$C_y = \{n \in \mathbb{Z} : n \geq 3,\ y - 1 \leq \log n \leq y,\\ \sigma(n) \geq e^{\gamma_E} n \log\log n\}.$$

This includes equality at the threshold and the finite small-n range. It does not assert that any candidate exists. Define the full relaxation

$$E_y(H) = \{n \in \mathbb{Z} : n \geq 1,\ y - 1 \leq \log n \leq y,\ B_\varphi(n) \leq H\}.$$

For ℓ in $[y-1, y]$, put

$$U_y(\ell) = \log M(y) - \gamma_E - \log(\log \ell) + \lambda(\ell - \theta(y)).$$

Here $\ell \geq 3$, so every logarithm is defined and $\log \ell > 0$. For $n \geq 3$ in the shell, taking logarithms of the positive Robin threshold gives exactly

$$n \in C_y \quad \Longleftrightarrow \quad B_\sigma(n) \leq U_y(\log n). \tag{I1}$$

The Euler product for σ gives $\sigma(n)/n \leq n/\varphi(n)$; hence $B_\varphi(n) \leq B_\sigma(n)$. Both benefits are nonnegative: for B_φ, omission below y costs $r(p) - \lambda \log p \geq 0$, first inclusion above y costs $\lambda \log p - r(p) \geq 0$, and each additional copy costs $\lambda \log p \geq 0$. These signs follow from the decrease of $r(t)/\log t$ for $t > 1$. Then use $B_\sigma \geq B_\varphi$. Therefore a uniform upper bound $U_y \leq H$ implies

$$C_y \subset \{\text{shell},\ B_\sigma \leq H\} \subset E_y(H). \tag{I2}$$

The first inclusion uses (I1), not merely the inequality between benefits.

4.2 Exact centered identity

Admit the classical Mertens normalization

$$\log M(T) - \log\log T \longrightarrow \gamma_E.$$

The source normalization is recorded in [Z] . Section 3 proves existence of the following improper integral under the theorem inputs:

$$I_\theta(y) = \int_y^\infty \frac{\Delta(t)(\log t + 1)}{t^2 \log^2 t}\, dt.$$

Put

$$R_{\text{tail}}(y) = \sum_{p > y} \sum_{j \geq 2} \frac{1}{jp^j} \geq 0,$$

$$S_1(y) = \sum_{p \leq y} \frac{1}{p}, \qquad Q_1(y) = \sum_{p \leq y} \sum_{j \geq 2} \frac{1}{jp^j}, \qquad k(t) = \frac{1}{t \log t}.$$

The total $Q_1(\infty)$ is finite by comparison with $\sum_{m \geq 2} 1/(m(m-1))$. Partial summation on $(y, T]$, with θ counting primes $\leq$ its argument, gives the exact finite-interval identity

$$S_1(T) - S_1(y) - \log\log T + \log\log y$$
$$= \Delta(T)k(T) - \Delta(y)k(y) + \int_y^T \frac{\Delta(t)(\log t + 1)}{t^2 \log^2 t}\, dt.$$

There is no half-weight convention at a prime endpoint. The atom at y is included in $\theta(y)$ and excluded from $(y, T]$; changing a single endpoint of the ordinary integral does not change its value.

For completeness, an elementary Chebyshev bound suffices for the boundary limit. For integers $m \geq 1$, the product of primes in $(m, 2m]$ divides binomial(2m,m), so $\theta(2m) - \theta(m) \leq 2m \log 2$. Summing at powers of two and then enclosing a real t in the next power of two gives

$$0 \leq \theta(t) \leq 4(\log 2)t, \qquad |\Delta(t)| \leq (4\log 2 + 1)t \quad (t \geq 2).$$

Thus $\Delta(T)k(T) \to 0$. Use $\log M = S_1 + Q_1$ and the Mertens limit above to pass to $T = \infty$, obtaining

$$\log M(y) - \gamma_E - \log\log y = \frac{\Delta(y)}{yu} - I_\theta(y) - R_{\text{tail}}(y).$$

Substitution into $U_y(y)$ yields

$$U_y(y) = \left(\frac{1}{yu} - \lambda\right)\Delta(y) - I_\theta(y) - R_{\text{tail}}(y). \tag{I3}$$

The $-R_{\text{tail}}$ sign is useful for an upper bound. No sign of I_θ is assumed.

4.3 Uniform arithmetic corrections

The convergent logarithm series, with $y \geq 4$, gives

$$0 \leq r(y) - \frac{1}{y} = \sum_{k\geq 2} \frac{1}{ky^k} \leq \frac{1}{2y(y-1)},$$
$$0 \leq \lambda - \frac{1}{yu} \leq \frac{1}{y^2 u}. \tag{I4}$$

Consequently the absolute endpoint contribution in (I3) is at most $c_\theta/(yu)$, where $c_\theta = 4\log 2 + 1$.

For t in $[y-1, y]$, $t \geq y/2$, $\log t \geq u/2$, and $\log t \geq \log 3 > 1$. Hence

$$|k'(t)| = \frac{\log t + 1}{t^2 \log^2 t} \leq \frac{16}{y^2 u}.$$

Differentiate the full threshold, keeping y fixed:

$$U_y'(\ell) = \lambda - k(\ell) = [\lambda - k(y)] + [k(y) - k(\ell)].$$

By (I4) and the mean value bound for k, uniformly on the whole closed shell,

$$|U_y'(\ell)| \leq \frac{17}{y^2 u}, \qquad |U_y(\ell) - U_y(y)| \leq \frac{17}{y^2 u}. \tag{I5}$$

Thus, including endpoints and allowing either sign of Δ and I_θ,

$$\sup_{\ell\in[y-1,y]} U_y(\ell) \le \max(0,-I_\theta(y)) + \frac{c_\theta}{yu} + \frac{17}{y^2u}. \tag{I6}$$

These explicit arithmetic constants are convenient majorants; they do not supply an explicit constant for the imported spectral bound or final count.

4.4 Positive-budget inclusion

Proposition 3. Let $A(y) > 0$ eventually, $yA(y) \to \infty$, and suppose for a fixed $C_I \ge 0$ that

$$\max(0,-I_\theta(y)) \le C_I A(y). \tag{I7}$$

for all sufficiently large y. Retain the normalization and integral existence conditions of Section 4.2. Choose any fixed $K_0 > C_I$ and set $H(y) = K_0A(y)$. Then eventually $\sup U_y \le H(y)$, and (I2) holds.

Proof. The two correction terms in (I6), divided by A, tend to zero:

$$\frac{c_\theta}{yuA} + \frac{17}{y^2uA} \longrightarrow 0.$$

Set $\epsilon = K_0 - C_I > 0$. Beyond one common threshold they sum to at most ϵA, and $(I6)$–$(I7)$ give $\sup U_y \le K_0A$. Combine with $(I1)$–$(I2)$. All constants are fixed independently of y. The threshold can depend on the inputs and the margin $K_0 - C_I$. QED.

The sufficient growth assumption $yA \to \infty$ is deliberately generous, not claimed necessary. Likewise (I7) is a sufficient one-sided input; the stronger absolute estimate $|I_\theta| \le C_I A$ also supplies it. No extra signed cancellation is needed once an absolute bound is available at the smaller spectral scale.

If $\sup U_y < 0$, nonnegative B_σ already makes C_y empty. If $\sup U_y = 0$, no emptiness assertion is needed: the same strictly positive H works. Never substitute the possibly zero positive-part allowance into (P1).

4.5 The single-shell consequence

Take $A(y) = P_0(y)\exp(-W(y))/u$ from Section 3 and fix $K_0 > C_I$. Set $H = K_0A$. The exact secondary factor satisfies $\log P_0 = o(W)$, and the theorem assumes $W = o(u), W/\log u \to \infty$. Therefore

$$\begin{aligned}
\log(yA) &= u + \log P_0 - W - \log u \longrightarrow \infty,\\
\log H &= \log K_0 + \log P_0 - W - \log u = o(u),\\
\log(Hu^3) &= \log K_0 + \log P_0 - W + 2\log u \longrightarrow -\infty.
\end{aligned}$$

The same positive H both bounds the full candidate threshold, by Proposition 3, and satisfies (P1). Proposition 2 consequently gives

$$\begin{aligned}
|C_y| \le |E_y(H)| &\le \exp(K_{\text{count}} y\sqrt{H/u})\\
&= \exp(K_{\text{count}}\sqrt{K_0}\,g(y)).
\end{aligned} \tag{SHELL}$$

K_{count} is absolute; K_0 depends on the fixed analytic inputs. Choose one threshold meeting all preceding eventual requirements. The construction of H, the inclusion and the count are separate deductions, and no local θ-error hypothesis has entered them.

5 Exact-rate regularity and global aggregation

5.1 A finite shell cover

Let C_y consist of integers $n \geq 3$ with $y - 1 \leq \log n \leq y$ and $\sigma(n) \geq \exp(\gamma_E) n \log\log n$. Let $R(X)$ count the same threshold candidates with $3 \leq n \leq X$, for real $X \geq 3$.

Suppose g is positive and nondecreasing eventually, $\log y = o(g(y))$, and there is a fixed $D_{\text{shift}} \geq 1$ such that

$$g(y + h) \leq D_{\text{shift}} g(y) \quad (0 \leq h \leq 1,\ y \text{ sufficiently large}).$$

Suppose also there are fixed $L > 0$ and y_0 such that

$$|C_y| \leq \exp(Lg(y)) \qquad \text{for every real } y \geq y_0. \tag{G1}$$

It suffices to assume (G1) at integer y. Then there is a fixed $L_{\text{global}} > 0$ such that eventually

$$R(X) \leq \exp(L_{\text{global}} g(\log X)). \tag{G2}$$

Proof. Fix an integer $J \geq 4$ beyond all eventual thresholds. Put $m = \text{ceil}(\log X)$ and $B_0 = \text{ceil}(\exp J)$, a fixed integer. A candidate with $\log n \leq J$ lies in the initial interval counted by B_0. Every remaining candidate belongs to C_j with $j = \text{ceil}(\log n)$, where $J < j \leq m$. At an integer logarithm the endpoint convention is still valid; overlapping shell endpoints only enlarge an upper count. Hence, for $m \geq J$,

$$\begin{aligned} R(X) &\leq B_0 + \sum_{j=J+1}^{m} |C_j| \\ &\leq B_0 + (m+1)e^{Lg(m)} \leq (B_0 + m + 1)e^{Lg(m)}. \end{aligned} \tag{G3}$$

The last step uses $\exp(Lg(m)) \geq 1$. With $y = \log X, m - y$ in $[0, 1)$, so

$$\log[(B_0 + m + 1)e^{Lg(m)}] \leq LD_{\text{shift}} g(y) + \log(B_0 + m + 1).$$

Since $m/y \to 1$ and $\log y = o(g(y))$, the final term is $o(g(y))$. For example $L_{\text{global}} = LD_{\text{shift}} + 1$ works after enlarging the threshold. This bounds $R(X)$ without taking log of $R(X)$, which may be zero. QED.

The number of shells is $O(\log X)$; the quantity absorbed in the exponent is its logarithm $O(\log\log X)$, not the number of shells itself. A finite initial segment never needs to satisfy the asymptotic shell hypothesis.

5.2 Regularity of the exact infimum

To separate variables, write

$$w(u) = W_2(e^u) = \inf_{t \geq T_*} \{\eta(t)u + 2\log t\}, \qquad T_* > 1.$$

Assume $0 < \eta(t) \leq 1/2$, with the established spectral limits

$$w(u) = o(u), \qquad \frac{w(u)}{\log u} \longrightarrow \infty. \tag{G4}$$

For the actual spectral secondary factor, with fixed $A_d > 0, B \geq 1, C \geq 0$, define

$$\begin{aligned} p(u) &= \left(\frac{u}{w(u)}\right)^{BC} \exp\left(\frac{A_d w(u)^{B+1}}{u^B}\right), \\ g(e^u) &= \frac{e^u \sqrt{p(u)} e^{-w(u)/2}}{u}. \end{aligned} \tag{G5}$$

All quantities are eventually positive, and $p \geq 1$ eventually. These are exact definitions, not asymptotic replacements inside an exponential.

For $0 < u \leq v$ the affine functions defining w have nonnegative slopes and positive intercepts. Taking infima gives

$$w(u) \leq w(v) \leq w(u) + (v-u)/2, \qquad w(v) \leq (v/u)w(u). \tag{G6}$$

For the second inequality, each $\eta(t)v + 2\log t$ is at most $(v/u)[\eta(t)u + 2\log t]$; choose arbitrarily close minimizers if necessary. Thus w is locally Lipschitz, hence locally absolutely continuous, and at almost every $u > 0$,

$$0 \leq w'(u) \leq w(u)/u. \tag{G7}$$

Indeed the right difference quotients are bounded by $w(u)/u$ using (G6). No unique minimizer, derivative of η, or everywhere differentiability of w is assumed. One could also note concavity of w, but it is not needed.

5.3 Regularity of the secondary factor

Put $b = BC$ and $q = w/u$. On each sufficiently large compact interval, w is positive; compositions in (G5) are locally absolutely continuous. At almost every u,

$$(\log p)' = b\left(\frac{1}{u} - \frac{w'}{w}\right) + A_d q^B[(B+1)w' - Bq]. \tag{G8}$$

By (G7), $0 \leq w' \leq q$. Therefore

$$\begin{gathered} 0 \leq b(1/u - w'/w) \leq b/u, \\ -A_d B q^{B+1} \leq A_d q^B[(B+1)w' - Bq] \leq A_d q^{B+1}. \end{gathered}$$

In particular

$$|(\log p)'| \le b/u + A_d B q^{B+1} \longrightarrow 0.$$

Here the last limit follows from $q = w/u \to 0$ and $B \ge 1$. Writing $L_{\text{rate}}(u) = \log(g(\exp u))$, (G5) gives

$$L'_{\text{rate}} = 1 - 1/u + (\log p)'/2 - w'/2.$$

Its distance from 1 is bounded almost everywhere by the deterministic quantity

$$1/u + b/(2u) + (A_d B/2) q^{B+1} + q/2 \longrightarrow 0.$$

Choose u_0 beyond which this bound is at most $1/2$. The fundamental theorem for absolutely continuous functions then gives, for $u_0 \le u \le v$,

$$\frac{v-u}{2} \le L_{\text{rate}}(v) - L_{\text{rate}}(u) \le \frac{3(v-u)}{2}. \tag{G9}$$

This justifies the monotonicity inference even at nondifferentiable points. In particular g is eventually increasing, and for $y \ge \exp u_0$, h in $[0,1]$,

$$1 \le \frac{g(y+h)}{g(y)} \le \left(\frac{y+h}{y}\right)^{3/2}.$$

One may use the fixed shift bound $D_{\text{shift}} = 3$ for $y \ge \max(1, \exp u_0)$, since $((y+h)/y)^{3/2} \le 2^{3/2} < 3$ there. Only this eventual range is used in Section 5.1.

Also

$$\log p = b \log(u/w) + A_d w (w/u)^B = o(w) = o(u).$$

The first term is $O(\log u) = o(w)$, since eventually $1 < w < u$. Thus $\log(g(\exp u)) = u + o(u)$, and $g(y) = y^{1+o(1)}$; in particular $\log y = o(g(y))$. All hypotheses of Section 5.1 are now proved for the exact factor (G5).

Scope. This proof covers the stated spectral infimum and specific secondary factor under (G4), not only smooth classical/VK shapes. Merely assuming $\log P_0 = o(W)$ for an arbitrary positive P_0 would not justify monotonicity or bounded shifts. Such a general budget needs separate rate regularity or a maximum-envelope version of (G3).

Completion of Theorem 1. Apply Section 5.1 to (SHELL) with $L = K_{\text{count}}\sqrt{K_0}$. Sections 5.2–5.3 establish positivity, monotonicity, bounded unit shifts and $\log y = o(g(y))$ for the exact theorem rate. Enlarge the fixed shell threshold once to include all analytic and counting thresholds. The finite initial segment is absorbed in (G3), and (T1) follows. No logarithm of an empty candidate count is taken.

6 Classical and Vinogradov–Korobov corollaries

For an admitted classical zero-free region $\eta(t) = a/\log t$ eventually, $a > 0$, the minimizer is eventually interior and elementary minimization gives

$$w(u) = 2\sqrt{2au}.$$

With the log-free density parameters $B = 1, C = 0, p = \exp(8A_d \text{ a})$ is a fixed constant. Consequently there are fixed L, X_0 such that $R(X) \le \exp(Lg(\log X))$ for $X \ge X_0$, where the global rate is

$$g(\log X) = \text{constant} \frac{\log X}{\log\log X} \exp(-\sqrt{2a \log\log X}). \tag{G10}$$

For an admitted VK region $\eta(\exp v) = av^{-2/3}(\log v)^{-1/3}$ eventually, retain the exact infimum $w(u)$. Only for bounding the secondary factor use

$$w(u) \asymp \frac{u^{3/5}}{(\log u)^{1/5}}.$$

Here is an order proof sufficient for that use. Let $v_0 = u^{3/5}/(\log u)^{1/5}$. Evaluating at v_0 bounds w above by a constant times v_0. For $v \ge v_0$ the term $2v$ gives a lower bound $2v_0$. For the remaining sufficiently large $v \le v_0$, the decreasing function $v^{-2/3}(\log v)^{-1/3}$ is at least its value at v_0, giving a positive constant times v_0. The fixed initial v interval has η bounded below by a positive constant and eventually contributes a much larger multiple of u. Thus w is comparable to v_0. This also verifies (G4).

With $B = 3/2, C = 15$,

$$\frac{w^{5/2}}{u^{3/2}} = O((\log u)^{-1/2}), \qquad \sqrt{p(u)} = O((u/w(u))^{45/4}).$$

Consequently (G2) implies the convenient upper bound with exponent scale

$$\frac{\log X}{\log\log X} \left(\frac{\log\log X}{W_2(\log X)}\right)^{45/4} \exp(-W_2(\log X)/2). \tag{G11}$$

Aggregation is first applied to the exact g in (G5); only afterward is $\sqrt{p}$ upper-bounded. No monotonicity of an arbitrary asymptotic replacement is needed, and w is never replaced inside $\exp(-w/2)$.

For a concrete classical instance, [MTY, Theorem 1.3] permits $a = 1/5.558691$, and the density estimate has $A_d = 5/2, B = 1, C = 0$. Jutila [Ju77, Theorem 1, $(1.7), p.46$] gives $N(\alpha, T, q) \ll_\epsilon (qT)^{(2+\epsilon)(1-\alpha)}$ uniformly for $4/5 \le \alpha \le 1$ and $T \ge 1$. Taking $q = 1$ and the fixed $\epsilon = 1/2$ bounds our positive-height count (a subset of his rectangle) by the required log-free contract with a fixed implied constant. The same parameters are recorded in [J] . For a VK instance, [MTY, Theorem 1.1] permits $a = 1/55.241$, paired with the recorded Ivić density $A_d = 1600, B = 3/2, C = 15$. Enlarge the fixed height cutoff to meet all hypotheses. These sufficient choices are not asserted to be optimal. The Ivić original statement has not been independently checked here; its uniform range $\sigma \ge 152/155$ and parameters are quoted through [J] . Independently, Bellotti [Be23, Theorem 1.2] gives a uniform estimate with $A_d = 57.8875, B = 3/2$ and $C = 10393/900$ for σ in $[0.98, 1]$ and $T \ge 3$. For $T \ge e$, this implies the conservative contract $A_d = 1600, B = 3/2, C = 15$ on $\sigma \ge 152/155$: both the T exponent and the logarithmic exponent are enlarged, and the finitely many displayed constants are bounded by one fixed constant, namely 2.15 times 10^{23}. For $T \le 3$ times 10^{12}, [Be23, $p.1$] records that the positive-height count in this strip is zero by the verified low-height zero computation. The constant therefore covers all $T \ge 3$; alternatively, the fixed cutoff T_* can simply be enlarged. Thus the existing VK specialization is also supported by a directly inspected primary statement, without changing (G11). Both specializations are unconditional consequences of these named ordinary-zeta theorems. Inspection of their statements is not a re-audit of their complete proofs.

7 Attribution, comparison and references

The counting argument is a short adaptation of classical benefit and partition methods [N69, EN75, ENS]. Benefit inequalities, moving benefit allowances and prime counting near a cutoff have earlier precedents; they are not claimed as new here. The proof also uses classical integrated explicit-formula and density estimates. It assembles their stated inputs into the candidate-counting theorem above. The manuscript does not claim that this composition, or its final bound, is new.

The corrected Luca–Pomerance–Solé theorem [LPS] bounds the corresponding exception count beyond 7! by $X^{O(1/\log\log X)}$. The finite convention change has no effect here. Relative to its exponent scale, the factor in our theorem is $\sqrt{P_0}\exp(-W/2)$, which tends to zero because $\log P_0 = o(W)$. This compares the displayed bounds only: it does not determine what sharper conclusions might already follow from their proof or other published results.

The conclusion of [LPS, Section 3] also shows that its coarse auxiliary set $S(x) = \{n \le x : \omega(n) > y\}, y = \log x/\log\log x$, has a lower bound $\exp((1+o(1))y)$, alongside its $\exp(O(y))$ upper bound. It discusses a refined threshold $y + y/\log\log x$. This limits the coarse relaxation, not every refinement of their method, and does not establish an exact asymptotic for $|S(x)|$.

The infimum framework is related to classical PNT error estimates: [J] credits Ingham [In32] with a weight-one infimum and Pintz [Pi80] with an improved error estimate. Our weight-two expression uses the quadratic zero denominator from integration. These connections are attribution context, not an assertion that those papers already prove the present counting theorem. Nicolas [N83] is also relevant to the $n/\varphi(n)$ relaxation. The full prior-implication analysis against these originals, [N69] , [EN75] and [R84] remains incomplete.

A completed bounded literature audit did not establish an exact prior statement or a demonstrated implication of the full theorem. This is not a novelty certificate. The Nicolas originals have now been recovered; their full implication analysis, and refined round-number counting behind LPS, remain open. The potential contribution is the spectral-budget derivation and its counting composition, subject to those comparisons. No effective numerical L or cutoff has been extracted. Unspecified constants do not establish inherent ineffectivity or vacuity at every concrete X. No lower bound for candidates, absence of exceptions, or RH conclusion follows from this upper bound.

[ENS] $P.$ Erdős, $J.-L.$ Nicolas and $A.$ Sárközy, On Large Values of the Divisor Function, Ramanujan Journal $2(1998)$, pp.225–245; Lemmas 1–2 and the near-prime argument adjoining Theorem 6, especially pp.238–242. Publisher chapter record . Primary article scan in the proceedings volume . Lemma 2, equation (69), printed $p.238$, is the one-variable cumulative prime-partition estimate used here. Its short proof invokes the exact prime-partition asymptotic of Hardy–Ramanujan and sums over $n \le x$; that upstream proof is not reproduced here. The full moving-budget integer theorem is not invoked.

[J] $D.R.$ Johnston, Zero-density estimates and the optimality of the error term in the prime number theorem, (2.2), discussion after (2.7) and (2.9). Version 1, arXiv:2411.13791v1 . The log-free Jutila and Ivić density instances are imported as explicitly recorded in this version. Version 2 uses a different VK density witness; the present sufficient parameters are intentionally pinned to $v1$ and are not asserted optimal. Their original complete proofs were not re-audited. Jutila's original statement and Bellotti's alternative uniform statement were checked for this revision as explained in Section 6; Ivić's original statement remains unchecked. Johnston's pointwise PNT conclusion and its smoothness/unique-minimizer conditions are not inputs here.

[K] $A.$ Kumchev, The Distribution of Prime Numbers, Chapter 2, Exercise 16, printed $p.38$, and the Brun–Titchmarsh exercises. Author lecture notes . The primitive identity is stated there for $\psi_1(x) = \sum_{n\le x}(x-n)\Lambda(n)$. Since $F = \psi_1 - x^2/2 + 2$ and $\zeta'(0)/\zeta(0) = \log(2\pi)$, it supplies the normalization in Section 3.2.

It is a located classical import, not a proof of the explicit formula newly supplied here.

[MTY] $M.J.$ Mossinghoff, $T.S.$ Trudgian and $A.$ Yang, Explicit zero-free regions for the Riemann ζ-function, Theorems 1.1 and 1.3. Author paper .

[Z] $T.$ Zhao, On the mean values of the error terms in Mertens' theorems, Sections 1.1 and 2. Author paper . The Mertens normalization is recorded there. Its related tail expression is not substituted for the primitive identity, and its RH-dependent sign conclusions are not imported. The inspected results do not state our unconditional first-power W_2/P_0 estimate; that estimate is derived in Section 3 here. This attribution does not assert that no equivalent estimate exists elsewhere.

[LPS] $F.$ Luca, $C.$ Pomerance and $P.$ Solé, corrected Theorem 1 on Robin exceptions, Corrigendum to $\#A8$ of Volume 25, On Robin's Inequality, Integers $25(2025)$, Theorem 1. Corrected paper .

[Po] $C.$ Pomerance, On the distribution of round numbers, Lecture Notes in Mathematics $1122(1985)$, pp.173–200, Theorem $6.1, p.195$. Author copy . This is the slice-count estimate used by LPS. At their thresholds its stated upper bound alone gives $\exp(O(\log X/\log\log X))$, not the smaller scale required here. This observation does not exclude refined thresholds or stronger information about those slices.

[N69] $J.-L.$ Nicolas, Ordre maximal d'un élément du groupe S_n des permutations et « highly composite numbers », Bulletin SMF $97(1969)$, pp.129–191, Chapter 3, especially pp.154 and 167. Primary archive .

[EN75] $P.$ Erdős and $J.-L.$ Nicolas, Répartition des nombres superabondants, Bulletin SMF $103(1975)$, pp.65–90, Proposition 5, pp.73–74. Primary archive .

[R84] $G.$ Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, $J.$ Math. Pures Appl. $(9)63(1984)$, pp.187–213, Theorem 1. The criterion is also stated in [LPS, Introduction]; the original full proof was not re-audited for this revision.

[Ra97] $S.$ Ramanujan, Highly composite numbers, annotated by $J.-L.$ Nicolas and $G.$ Robin, Ramanujan Journal $1(1997)$, pp.119–153, Section 71 and note 71. Historical context; no additional theorem from this source is used as an input.

[N83] $J.-L.$ Nicolas, Petites valeurs de la fonction d'Euler, Journal of Number Theory $17(1983)$, pp.375–388. Background on the related totient criterion; its implications for this counting theorem remain to be analyzed.

[In32] $A.E.$ Ingham, The Distribution of Prime Numbers, Cambridge University Press, 1932, Theorem 22 (as discussed in [J]).

[Pi80] $J.$ Pintz, On the remainder term of the prime number formula II; On a theorem of Ingham, Acta Arithmetica $37(1980)$, pp.209–220. Historical attribution here follows [J] ; its original proof is not an input.

[Ju77] $M.$ Jutila, On Linnik's constant, Mathematica Scandinavica $41(1977)$, pp.45–62, Theorem $1, (1.7)$, printed $p.46$. Primary journal scan . The uniform statement was inspected directly; Section 6 explains the specialization.

[Iv] $A.$ Ivić, The Riemann Zeta-Function, Wiley, 1985; Dover reprint 2003, Theorem 11.3, as quoted by [J] . Original statement and proof not directly verified here.

[Be23] $C.$ Bellotti, Explicit zero density estimate near unity, arXiv:2311.05136v1 (2023), Theorem 1.2, pp.2–3. Primary paper . Its uniform statement implies the conservative VK density contract retained here.

[GM] $L.$ Grenié and $G.$ Molteni, Explicit smoothed prime ideals theorems under GRH, Math. Comp. $85(2016)$, pp.1875–1899; $(1.3a), (3.8)$, Lemma 3.3. Author version . The smoothed identity used here is unconditional; their later GRH bounds are not used.

[MV73] $H.L.$ Montgomery and $R.C.$ Vaughan, The large sieve, Mathematika $20(1973)$, pp.119–134, doi:10.1112/S0025579300004708. Primary attribution for Brun–Titchmarsh with constant 2. The exact bound in Section 2.3 is retained as a named classical import. [K, printed $p.73$, after Exercise 7] records the sharp progression bound with $N > q$. The original MV73 theorem number and its precise real-endpoint convention have not been directly checked here: the accessible primary PDF routes failed. This source-check limitation does not change the stated input contract or assert a defect in the bound.

[Me74] $F.$ Mertens, Ein Beitrag zur analytischen Zahlentheorie, $J.$ reine angew. Math. $78(1874)$, pp.46–62. Historical attribution; the normalization used here is located in [Z] .

8 Development, verification and scope

The written proof in Sections 1–6 does not depend on a Lean implementation or on unpublished auxiliary notes. Selected finite encoding, benefit-splitting, outside-coordinate and distance-budget steps have companion Lean checks; the finite partition interface is conditional on Brun–Titchmarsh and ENS Lemma 1. The ENS partition asymptotic, spectral tail, centered inclusion and complete analytic specialization are not formally verified by that development. These are unpublished internal checks, not a publicly reproducible formal-verification supplement or formal verification of Theorem 1 or RH.

Codex assisted with proof development, implementation and manuscript maintenance; user-run model sessions supplied adversarial reviews. Such AI-assisted development and checking do not constitute human peer review or certify the assembled manuscript. Specialist review of correctness and attribution remains outstanding. The exact classical inputs and source-check limits are stated in Sections $1, 2.3, 3.2, 6$ and 7.

This revision concerns the September analytic manuscript only. Finite canonical certification over real y in $[100000, 102000]$ is a separate project and is not evidence for this theorem. Neither that finite certification nor this asymptotic upper bound establishes RH or a global sign result.